On complex singularities of the 2D Euler equation at short times

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We present a study of complex singularities of a two-parameter family of solutions for the two-dimensional Euler equation with periodic boundary conditions and initial conditions $\psi_0(z_1, z_2) = \hat{F}(\mathfrak{p}) \cos \mathfrak{p} \cdot \boldsymbol{z} + \hat{F}(\mathfrak{q}) \cos \mathfrak{q} \cdot \boldsymbol{z}$ in the short-time asymptotic régime. As has been shown numerically in W. Pauls *et al.*, Physica D**219**, 40–59 (2006), the type of the singularities depends on the angle ϕ between the modes $\mathfrak{p}$ and $\mathfrak{q}$. Here we show for the two particular cases of ϕ going to zero and to π that the type of the singularities can be determined very accurately, being characterised by $\alpha = 5/2$ and $\alpha = 3$ respectively. In these two cases we are also able to determine the subdominant corrections. Furthermore, we find that the geometry of the singularities in these two cases is completely different, the singular manifold being located "over" different points in the real domain.

I. INTRODUCTION

One of the most important open problems in mathematical fluid dynamics is the question of well-posedness of the three-dimensional incompressible Navier–Stokes and Euler equations (for the most recent reviews see [1, 2, 3]). Although the Navier–Stokes equation is more relevant from the physical point of view, certain phenomena related to the non-locality of the equations are partially obscured by the presence of the viscous term and are more conveniently studied in the inviscid case. One of these phenomena is the *depletion* of nonlinearity: generally flows behave much more tamely than one would expect based on simple dimensional arguments and tend to suppress singularities, approaching – at least locally – one of the numerous steady state solutions of the Euler equation.

Typically, inviscid incompressible flows observed in numerical simulations exhibit a much slower loss of regularity than that predicted by analytical estimates. However, it should be taken into account that numerical simulations aiming at studying the blow-up are mostly performed on periodic domains by employing the pseudo-spectral method with very smooth initial conditions, such as trigonometric polynomials. In contrast to numerical studies, analytical estimates aim at much larger classes of initial conditions for which the rate of loss of regularity may be much faster. Indeed, in two dimensions Bahouri and Chemin [4] have constructed an example of a vortex patch initial condition with exponentially fast regularity loss, thus proving that the corresponding estimate of the regularity loss is sharp. Therefore, it is important to clarify how the strength of depletion may depend on the character of the initial conditions.

One possibility to assess the strength of depletion and its dependence on the initial conditions consists in analysing complex singularities[1] of solutions of the Euler equation: for real analytic initial conditions, solutions can be continued into the complex domain by extending the space variables to complex values. These solutions typicaly become singular at some complex locations constituting, according to the numerical evidence given in [6, 7], a smooth manifold. Depletion is especially pronounced in the neighbourhood of the complex singularities: assuming that the singular manifold has co-dimension one, it follows from the incompressibility condition that nonlinearity vanishes to the leading order in the direction normal to it.

It has been found in [8] (in the following abbreviated as PMFB) by high-precision numerical simulations that this proliferation of depletion in the neighbourhood of singularities entails a significant dependence of the analytic properties of solutions on the initial conditions: it affects the geometry but also the type of the singularities. The singularities are non-universal: strong numerical evidence has been given in PMFB that in the neighbourhood of the singularities the vorticity behaves like $s^{-\beta}$ (here s is the distance to the singular manifold) with the exponent β depending on the choice of the initial condition.

Let us now briefly describe some of the findings in PMFB. In that work, solutions corresponding to the initial conditions $\psi_0(z_1, z_2) = \hat{F}(\mathfrak{p}) \cos \mathfrak{p} \cdot \boldsymbol{z} + \hat{F}(\mathfrak{q}) \cos \mathfrak{q} \cdot \boldsymbol{z}$ were considered in the short-time asymptotic régime. The exponent β was determined indirectly by analysing the asymptotic behaviour of the Fourier coefficients of the stream function along so-called diagonals (i.e. directions with rational slope) in the Fourier space at high wavenumbers. Asymptotically, the Fourier coefficients were found to behave as product of an exponential (which gives the leading order contribution) with an algebraic prefactor $|\mathbf{k}|^{-\alpha}$ (being the first subdominant contribution). The exponent α is related to the rate of divergence of the vorticity β via $\alpha + \beta = 7/2$ and can be determined numerically. The precision in determining the numerical value

[1] An additional reason for studying complex singularities is that possible emergence of real singularities in three dimensions can be detected by monitoring the dynamics of complex singularities [12], since, as has been shown by Bardos et al [9] and Benachour [10, 11], a hypothetic real singularity can only arise when a singularity located in the complex domain hits the real domain.

of the exponent α was sufficient to insure that the type of the singularities changes depending on the initial conditions: two different initial conditions were found with $\alpha = 2.66 \pm 0.01$ and $\alpha = 2.54 \pm 0.01$. It was also found that the numerical value of the algebraic prefactor exponent α depends on the angle between the vectors $\mathfrak{p}$ and $\mathfrak{q}$, but does not depend on their lengths.

To achieve a better understanding of the non-universality of the type of complex singularities and of their dependence on the initial conditions a systematic study of the numerical values of α and their dependence on the angle between $\mathfrak{p}$ and $\mathfrak{q}$ is needed. However, such a study requires a more precise determination of the values of the exponent α as a function of the initial data, since for small values of the angle between $\mathfrak{p}$ and $\mathfrak{q}$ the changes in the numerical value of α are of the same order as the errors in α.

Unfortunately, for the cases which have been presented in PMFB a more precise determination of the exponent is not feasible with the resolution achieved there. The main reason for this is the presence of strong intermediate asymptotics and our lack of knowledge of the asymptotic expansion of the Fourier coefficients along diagonals in the Fourier space.

Here we shall show that there exist initial conditions for which it is possible to achieve high precision for the numerical value of α and to determine the subdominant contributions: in the one case we find $\alpha = 5/2$, in the other $\alpha = 3$. In Section II we introduced a rescaled form of the two-dimensional Euler equation, describe Taylor expansions of its solutions, their relation to complex singularities and to the short-time behaviour of solutions. In Section III we give a detailed account of the short-time asymptotic régime, in particular with respect to the dependence on the angle between the vectors $\mathfrak{p}$ and $\mathfrak{q}$. In Section IV we study the case of initial conditions with the angle between $\mathfrak{p}$ and $\mathfrak{q}$ going to zero. We show that the numerical value and the asymptotic expansion of the Fourier coefficients can be determined very accurately. In Section V we study the limiting case of the vectors $\mathfrak{p}$ and $\mathfrak{q}$ becoming anti-parallel. We show that in contrary to the cases studied previously in PMFB the complex singularities are not located "above" a symmetry point. We also determine accurately the value of α and the asymptotic expansion of the Fourier coefficients. In Section VI we make some remarks on the general case, when the angle between $\mathfrak{p}$ and $\mathfrak{q}$ is not restricted to any special value. Finally, we conclude with Section VII.

II. RESCALED TWO-DIMENSIONAL EULER EQUATION

A. Basic equation

We begin by considering the two-dimensional Euler equation

$$\partial_t \Delta\psi = J(\psi, \Delta\psi), \tag{1}$$

on the periodic domain $\mathbb{T}^2_{\mathbb{R}} = [0, 2\pi) \times [0, 2\pi)$. We chose the simplest periodic initial conditions with nontrivial dynamics, that is, trigonometric polynomials with only two harmonics

$$\psi_0(z_1, z_2) = \hat{F}(\mathfrak{p}) e^{-i\mathfrak{p}\cdot\boldsymbol{z}} + \hat{F}(\mathfrak{q}) e^{-i\mathfrak{q}\cdot\boldsymbol{z}} + c.c. \tag{2}$$

Note that after a real translation we can assume that the initial amplitudes are real. Therefore the initial condition becomes

$$\psi_0(z_1, z_2) = 2\,\hat{F}(\mathfrak{p}) \cos \mathfrak{p}\cdot\boldsymbol{z} + 2\,\hat{F}(\mathfrak{q}) \cos \mathfrak{q}\cdot\boldsymbol{z}. \tag{3}$$

Since we shall restrict ourselves to the initial conditions (3), we introduce the auxiliary coordinates $\overline{z}_1 = \mathfrak{p}\cdot\boldsymbol{z}$, $\overline{z}_2 = \mathfrak{q}\cdot\boldsymbol{z}$ and a rescaled time $\tau = (\mathfrak{p}\wedge\mathfrak{q})t$. Then, writing the solution as

$$\psi(z_1, z_2; t) = \overline{\psi}(\overline{z}_1, \overline{z}_2; (\mathfrak{p}\wedge\mathfrak{q})\, t), \tag{4}$$

we obtain the Euler equation in the following form

$$\partial_\tau \overline{\Delta}_{(\mathfrak{p},\,\mathfrak{q})} \overline{\psi} = \overline{J}(\overline{\psi}, \overline{\Delta}_{(\mathfrak{p},\,\mathfrak{q})} \overline{\psi}), \tag{5}$$

with initial condition

$$\overline{\psi}_0(\overline{z}_1, \overline{z}_2) = 2\,\hat{F}(\mathfrak{p}) \cos \overline{z}_1 + 2\,\hat{F}(\mathfrak{q}) \cos \overline{z}_2, \tag{6}$$

where $\overline{\Delta}_{(\mathfrak{p},\,\mathfrak{q})}$ denotes the Lapacian

$$\overline{\Delta}_{(\mathfrak{p},\,\mathfrak{q})} = |\mathfrak{p}|^2\, \overline{\partial}_1^2 + 2\,\mathfrak{p}\cdot\mathfrak{q}\, \overline{\partial}_1 \overline{\partial}_2 + |\mathfrak{q}|^2\, \overline{\partial}_2^2. \tag{7}$$

Note that in this formulation the vectors $\mathfrak{p}$ and $\mathfrak{q}$ do not necessarily have to be vectors in $\mathbb{Z}^2$, but can take any arbitrary values in $\mathbb{R}^2$.

It is obvious that if $\overline{\psi}(\overline{z}_1, \overline{z}_2; t)$ is a solution of (4) with intitial condition $\overline{\psi}_0$, then $\overline{\psi}_\lambda(\overline{z}_1, \overline{z}_2; \tau_\lambda) = \lambda\overline{\psi}_\lambda(\overline{z}_1, \overline{z}_2; \lambda\tau)$ is the solution of the same equation with rescaled time $\tau_\lambda = \lambda\tau$ and the initial condition $\lambda\overline{\psi}_0$. Thus, the only relevant parameters characterising the problem are the ratio $\lambda = \hat{F}(\mathfrak{q})/\hat{F}(\mathfrak{p})$ of the two initial amplitudes $\hat{F}(\mathfrak{p})$ and $\hat{F}(\mathfrak{q})$ and the angle ϕ between the vectors $\mathfrak{p}$ and $\mathfrak{q}$ together with the ratio of lenghts $\eta = |\mathfrak{q}|/|\mathfrak{p}|$ of these two vectors. The latter fact is immediately seen by rewriting the Laplacian as

$$\overline{\Delta}_{(\mathfrak{p},\,\mathfrak{q})} = |\mathfrak{p}|\, \overline{\Delta}_{(\phi,\,\eta)}, \tag{8}$$

and obtaining

$$\partial_\tau \overline{\Delta}_{(\phi,\,\eta)} \overline{\psi} = \overline{J}(\overline{\psi}, \overline{\Delta}_{(\phi,\,\eta)} \overline{\psi}), \tag{9}$$

where

$$\overline{\Delta}_{(\phi,\,\eta)} = \frac{1}{\eta}\, \overline{\partial}_1^2 + 2\, \cos\phi\, \overline{\partial}_1 \overline{\partial}_2 + \eta\, \overline{\partial}_2^2. \tag{10}$$

B. Perturbative approach

From (10) it follows immediately that in the case $\eta = 1$ Equation (9) has a stationary solution. Since, as we have mentioned, the parameter η can in principle take any values, for values of η close to 1 we try a perturbative expansion in

$\varepsilon = \eta - 1$. The leading order $\overline{\psi}_0$ coincides with the initial condition

$$\overline{\psi}_0(z_1, z_2) = 2\,\hat{F}(\mathfrak{p})\cos\overline{z}_1 + 2\,\hat{F}(\mathfrak{q})\cos\overline{z}_2, \qquad (11)$$

whereas the next order $\overline{\psi}_1$ is the solution of the linearised Euler equation with a source term

$$\partial_t \Delta_\phi \overline{\psi}_1 = J(\overline{\psi}_0, \Delta_\phi \overline{\psi}_1) + J(\overline{\psi}_1, \Delta_\phi \overline{\psi}_0) + 2\,\hat{F}(\mathfrak{p})\,\hat{F}(\mathfrak{q})\,\sin z_1\,\sin z_2. \qquad (12)$$

The background flow $\overline{\psi}_0$ is the well-known cat's eye flow. Note that neglecting the source term and the perturbation term $J(\overline{\psi}_1, \Delta_\phi \overline{\psi}_0)$ we obtain the shortened Euler equation which is just the passive scalar equation

$$\partial_t\, \Delta_\phi \overline{\psi}_1 + \boldsymbol{v} \cdot \nabla(\Delta_\phi \overline{\psi}_1) = 0, \qquad (13)$$

where the velocity field is determined by $\boldsymbol{v} = (\overline{\partial}_2 \overline{\psi}_0, -\overline{\partial}_1 \overline{\psi}_0)$. Singularities of advected fields such as $\Delta_\phi \overline{\psi}_1$ have been studied in [15, 16]. Since these singularities are determined by the singularities of the back-to-labels map and the flow is integrable, the singularities can be determined completely, see [15] for the case $\lambda = 1$ ($\hat{F}(\mathfrak{p}) = \hat{F}(\mathfrak{q})$) and [16] for the more general case $0 \le \lambda \le 1$.

Coming back to the linearised Euler equation we note that it has been extensively studied for various types of background flows. In particular, in [17] flows fo Kolmogorov's type have been studied (this type of flows, however, does not have any complex singularities). However, an analytic study of singularities of solutions of the linearised time-dependent Euler equation has not been attempted here, but certainly merits a more detailed consideration.

C. Taylor series expansion

To calculate the solutions of Equation (9) we use the approach based on Taylor series expansion [21]: representing the solution as Taylor series in time

$$\overline{\psi}(\overline{\boldsymbol{z}}, t) = \sum_{n=0}^{\infty} \overline{\psi}^{(n)}(\overline{z})\tau^n, \qquad (14)$$

it is obvious that the functions $\overline{\psi}^{(n)}(\overline{z})$ satisfy the following recursion relation

$$\overline{\Delta}_{(\mathfrak{p},\mathfrak{q})}\overline{\psi}^{(n+1)} = \frac{1}{n+1}\sum_{m=0}^{n} \overline{J}(\overline{\psi}^{(m)}, \overline{\Delta}_{(\mathfrak{p},\mathfrak{q})}\overline{\psi}^{(n-m)}). \quad (15)$$

For example, for $\overline{\psi}^{(1)}$ we obtain

$$\overline{\psi}^{(1)}(\overline{z}) = 2\,\hat{F}(\mathfrak{p})\hat{F}(\mathfrak{q})\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}-\mathfrak{q}|^2}\cos(\overline{z}_1 - \overline{z}_2) - 2\,\hat{F}(\mathfrak{p})\hat{F}(\mathfrak{q})\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}+\mathfrak{q}|^2}\cos(\overline{z}_1 + \overline{z}_2). \qquad (16)$$

In general the functions $\overline{\psi}^{(n)}(\overline{z}_1, \overline{z}_2)$ will have the form

$$\begin{aligned}
\overline{\psi}^{(n)}(\overline{z}_1, \overline{z}_2) = &\sum_{m=0}^{2m<n}\sum_{\sigma=1}^{n-2m} 2\big(\hat{F}(\mathfrak{p})\big)^{n+1-2m-\sigma}\big(\hat{F}(\mathfrak{q})\big)^{2m+\sigma} \\
&\Big\{F_{+\ \sigma}^{(n,m)}\cos\big((n+1-2m-\sigma)\overline{z}_1 + \sigma\overline{z}_2\big) + \\
&F_{-\ \sigma}^{(n,m)}\cos\big((n+1-2m-\sigma)\overline{z}_1 - \sigma\overline{z}_2\big)\Big\} + \\
&\sum_{m=1}^{2m\le n} 2\big(\hat{F}(\mathfrak{p})\big)^{n+1-2m}\big(\hat{F}(\mathfrak{q})\big)^{2m}F_1^{(n,m)}\cos\big((n+1-2m)\overline{z}_1\big) + \\
&\sum_{m=1}^{2m\le n} 2\big(\hat{F}(\mathfrak{p})\big)^{2m}\big(\hat{F}(\mathfrak{q})\big)^{n+1-2m}F_2^{(n,m)}\cos\big((n+1-2m)\overline{z}_2\big).
\end{aligned} \qquad (17)$$

The coefficients $F_{+\ \sigma}^{(n,m)}$, $F_{-\ \sigma}^{(n,m)}$, $F_1^{(n,m)}$ and $F_2^{(n,m)}$ depend only on the initial vectors $\mathfrak{p}$ and $\mathfrak{q}$, more precisely on the parameters ϕ and η. For example, for $\overline{\psi}^{(2)}$ we have

$$\begin{aligned}
F_{+\ 1}^{(2,0)} &= \frac{1}{2}\,\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}+\mathfrak{q}|^2}\,\frac{|\mathfrak{p}+\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|2\mathfrak{p}+\mathfrak{q}|^2}, \\
F_{-\ 1}^{(2,0)} &= \frac{1}{2}\,\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}-\mathfrak{q}|^2}\,\frac{|\mathfrak{p}-\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|2\mathfrak{p}-\mathfrak{q}|^2}, \\
F_{+\ 2}^{(2,0)} &= -\frac{1}{2}\,\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}+\mathfrak{q}|^2}\,\frac{|\mathfrak{p}+\mathfrak{q}|^2 - |\mathfrak{q}|^2}{|\mathfrak{p}+2\mathfrak{q}|^2}, \\
F_{-\ 2}^{(2,0)} &= -\frac{1}{2}\,\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}-\mathfrak{q}|^2}\,\frac{|\mathfrak{p}-\mathfrak{q}|^2 - |\mathfrak{q}|^2}{|\mathfrak{p}-2\mathfrak{q}|^2}, \\
F_1^{(2,1)} &= \frac{1}{2}\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{p}|^2}\left(2 - \frac{|\mathfrak{q}|^2}{|\mathfrak{p}+\mathfrak{q}|^2} - \frac{|\mathfrak{q}|^2}{|\mathfrak{p}-\mathfrak{q}|^2}\right), \\
F_2^{(2,1)} &= -\frac{1}{2}\frac{|\mathfrak{q}|^2 - |\mathfrak{p}|^2}{|\mathfrak{q}|^2}\left(2 - \frac{|\mathfrak{p}|^2}{|\mathfrak{p}+\mathfrak{q}|^2} - \frac{|\mathfrak{p}|^2}{|\mathfrak{p}-\mathfrak{q}|^2}\right).
\end{aligned} \qquad (18)$$

Inserting the representation (17) into Equation (15) we obtain a system of recursion relations for the coefficients. Since the resutling expressions turn out to be quite bulky we prefer not to list them here (see, however, [22] in a slightly different notation).

Note that contrary to solutions of the original Euler equation the higher-order terms $\overline{\psi}^{(n)}(\overline{z}_1, \overline{z}_2)$ do not vanish trivially for $\phi \to 0$ (or $\phi \to \pi$) if we take the limit keeping the ratio η of lengths of $\mathfrak{p}$ and $\mathfrak{q}$ fixed. However, it is not obvious that the limit $\phi \to 0$ (or $\phi \to \pi$) is well-defined. Indeed, if we try to evaluate expressions (18) by putting directly, say $\phi = 0$ and $\eta = 1$ or $\eta = 2$, we obtain ill-defined expressions because of vanishing denominators. Thus, we have to be careful about specifying the limit $\phi \to 0$. Actually we are considering a double limit: the limit of the initial modes becoming parallel and the limit of long times, as can be seen from (4). We shall come back to this question in Section III.

Let us finally make a small remark on the utility of the time Taylor series expansion: introducing the auxiliary coordinates $\xi_1 = e^{-i\overline{z}_1}$ and $\xi_2 = e^{-i\overline{z}_2}$ we can write the solution in form of a triple Laurent series in τ, ξ_1 and ξ_2. The

solution obtained in this form is only meaningful inside the domain of convergence of the corresponding series[2]. The domain of convergence is determined by the complex singularities of the function $\overline{\psi}(\xi_1, \xi_2; \tau)$: in particular, because of the regularity of two-dimensional flows we know that for real $\overline{z}_1$, $\overline{z}_2$ there are no singularities for real values of τ. However, there will typically be singularities for some complex-valued $\tau^\star$, which will prevent the series from convergence even for real-valued τ's, when $\tau > |\tau^\star|$, see e.g. [23]. Thus, the power series representation is not well-suited for studying the temporal dynamics of the solution. Nevertheless, it lends itself very well for analysing space-time structure of the singularities. Indeed, it is known that from the asymptotic behaviour of the coefficients of a series conclusions can be made concerning the geometry and the type of the complex singularities of $\overline{\psi}(\xi_1, \xi_2; \tau)$, see [8, 22].

D. Short-time asymptotics

The calculation of the solution is greatly simplified if we consider the asymptotic short-time behaviour as in [6, 7, 8]. Indeed, at short times the Fourier coefficients corresponding to each $\cos(k_1\overline{z}_1 + k_2\overline{z}_2)$ can be represented as

$$\tau^{k_1+k_2-1} 2\big(\hat{F}(\mathfrak{p})\big)^{k_1}\big(\hat{F}(\mathfrak{q})\big)^{k_2} F_{+\ k_2}^{(k_1+k_2-1,0)}(1+O(\tau^2)), \tag{19}$$

for $k_2 > 0$ and

$$\tau^{k_1-k_2-1} 2\big(\hat{F}(\mathfrak{p})\big)^{k_1}\big(\hat{F}(\mathfrak{q})\big)^{-k_2} F_{-\ -k_2}^{(k_1-k_2-1,0)}(1+O(\tau^2)), \tag{20}$$

for $k_2 < 0$. The wave vector k_1 can be assumed positive without any loss of generality. Then, in the asymptotic régime $t \to 0$ the solution will have the form

$$\begin{aligned}&\overline{\psi}_{\rm as}(\overline{z}_1, \overline{z}_2) = \sum_{n=0}^{\infty}\sum_{\sigma=1}^{n} 2\big(\hat{F}(\mathfrak{p})\big)^{n+1-\sigma}\big(\hat{F}(\mathfrak{q})\big)^{\sigma}\tau^n \times \\ &\Big\{F_{+\ \sigma}^{(n,0)}\cos\big((n+1-\sigma)\overline{z}_1 + \sigma\overline{z}_2\big) + \\ &F_{-\ \sigma}^{(n,0)}\cos\big((n+1-\sigma)\overline{z}_1 - \sigma\overline{z}_2\big)\Big\}\end{aligned} \tag{21}$$

Representing $\cos\big((n+1-\sigma)\overline{z}_1 + \sigma\overline{z}_2$ and $\cos\big((n+1-\sigma)\overline{z}_1 - \sigma\overline{z}_2$ as a sum of two exponentials we can write the short-time asymptotic solution as

$$\begin{aligned}&\overline{\psi}_{\rm as}(\overline{z}_1, \overline{z}_2) = \frac{1}{\tau}\overline{\psi}_+(\overline{z}_1 + i\ln\tau, \overline{z}_2 + i\ln\tau) + \\ &\frac{1}{\tau}\overline{\psi}_-(\overline{z}_1 + i\ln\tau, \overline{z}_2 - i\ln\tau) + c.c.,\end{aligned} \tag{22}$$

[2] Unless we perform analytic continuation, which is the subject of an ongoing work.

where

$$\begin{aligned}&\overline{\psi}_+(\tilde{z}_1, \tilde{z}_2) = \\ &\sum_{n=0}^{\infty}\sum_{\sigma=1}^{n}\big(\hat{F}(\mathfrak{p})\big)^{n+1-\sigma}\big(\hat{F}(\mathfrak{q})\big)^{\sigma} F_{+\ \sigma}^{(n,0)} e^{-i[(n+1-\sigma)\tilde{z}_1 + \sigma\tilde{z}_2]}\end{aligned} \tag{23}$$

and, similarly

$$\begin{aligned}&\overline{\psi}_-(\tilde{z}_1, \tilde{z}_2) = \\ &\sum_{n=0}^{\infty}\sum_{\sigma=1}^{n}\big(\hat{F}(\mathfrak{p})\big)^{n+1-\sigma}\big(\hat{F}(\mathfrak{q})\big)^{\sigma} F_{-\ \sigma}^{(n,0)} e^{-i[(n+1-\sigma)\tilde{z}_1 - \sigma\tilde{z}_2]}.\end{aligned} \tag{24}$$

As has been noted in [8], the functions

$$\frac{1}{\tau}\,\overline{\psi}_+(\overline{z}_1 + i\ln\tau, \overline{z}_2 + i\ln\tau),\ \ \frac{1}{\tau}\,\overline{\psi}_-(\overline{z}_1 + i\ln\tau, \overline{z}_2 - i\ln\tau)$$

are actually solutions of the Euler equation (4) corresponding to initial conditions

$$\hat{F}(\mathfrak{p})e^{-iz_1} + \hat{F}(\mathfrak{q})e^{-iz_2} \tag{25}$$

and $\hat{F}(\mathfrak{p})e^{-iz_1} + \hat{F}(\mathfrak{q})e^{iz_2}$. Thus, at short times, the solution of the Euler equation (4) with initial condition (6) behaves as a linear superposition of four solutions with very simple two-mode initial conditions of the type (25)

Geometrically, the above observation can be explained as follows (see Fig. 1): the singular manifold consists of four branches which at short-times are located very far away from each other. Therefore, at short times every branch behaves as if it would not see the presence of other branches; the nonlinear interactions between separate branches are weak.

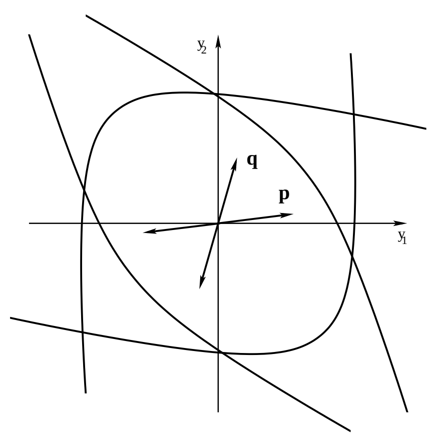

FIG. 1: Schematic representation of the projection of the singular manifold onto the (y_1, y_2)-plane at short-times. Only the parts of the singular manifold closest to the real domain are sketched. The singularities are represented in non-rescaled coordinates. Note that the vectors $\mathfrak{p}$ and $\mathfrak{q}$ determine the asymptotic directions of the various branches.

III. SOLUTIONS OF THE EULER EQUATION IN THE SHORT-TIME ASYMPTOTIC RÉGIME

A. Fourier space and pseudo-hydrodynamic formulations

In this section we study short-time asymptotic solutions of the two-dimensional Euler equation (4). Since the coefficients $F_{-\ \sigma}^{(n,0)}$ can be determined using the formula

$$F_{-\ \sigma}^{(n,0)}(\mathfrak{p},\mathfrak{q}) = (-1)^n F_{+\ \sigma}^{(n,0)}(\mathfrak{p},-\mathfrak{q}), \tag{26}$$

we concentrate on the solution given by $\overline{\psi}_+$. Here we assume that $\phi \neq 0$. In what follows we put for brevity $F_{+\ \sigma}^{(n,0)} = G_\sigma^{(n)}$. The coefficients $G_\sigma^{(n)}$ can be calculated by means of the following recursion relation

$$\begin{aligned} G_\sigma^{(n)} = &-\frac{1}{n}\frac{1}{|(n+1-\sigma)\mathfrak{p}+\sigma\mathfrak{q}|^2}\sum_{m=0}^{n-1}\sum_{\tau=0}^{\sigma} \\ &|((n-m)-(\sigma-\tau))\mathfrak{p}+(\sigma-\tau)\mathfrak{q}|^2 \\ &[(m+1)(\sigma-\tau)-(n-m)\tau]G_\tau^{(m)}G_{\sigma-\tau}^{(n-m-1)} \end{aligned} \tag{27}$$

with initial values $G_0^{(0)} = 1 = G_1^{(0)}$. It holds[3]

$$G_{n+1-\sigma}^{(n)}(\mathfrak{p},\mathfrak{q}) = (-1)^n G_\sigma^{(n)}(\mathfrak{q},\mathfrak{p}) \tag{28}$$

Similarly to what has been noted in PMFB06, the coefficients $G_1^{(n)}$ can be obtained explicitly

$$G_1^{(n)} = (-1)^n\frac{1}{n!}\prod_{\sigma=0}^{n-1}\frac{|\sigma\mathfrak{p}+\mathfrak{q}|^2-|\mathfrak{p}|^2}{|(\sigma+1)\mathfrak{p}+\mathfrak{q}|^2}, \tag{29}$$

as well as the coefficients $G_n^{(n)}$.

Analogously to PMFB the short-time asymptotics can be formulated as a stationary hydrodynamical problem by introducing the pseudo-hydrodynamic stream function

$$G(\tilde{z}_1,\tilde{z}_2) = e^{\tilde{z}_1}+e^{\tilde{z}_2}+\sum_{n=1}^{\infty}\sum_{\sigma=1}^{n}G_\sigma^{(n)}e^{(n+1-\sigma)\tilde{z}_1}e^{\sigma\tilde{z}_2}. \tag{30}$$

In terms of the pseudo-hydrodynamic stream function $G(\tilde{z}_1,\tilde{z}_2)$ the solution of the Euler equation (4) with the initial condition $\hat{F}(\mathfrak{p})e^{-iz_1}+\hat{F}(\mathfrak{q})e^{-iz_2}$ can be written as

$$\begin{aligned} &\frac{1}{t}\,\overline{\psi}_+ = \\ &\frac{1}{t}\,G(-i\overline{z}_1+\ln t+\ln F(\mathfrak{p}),-i\overline{z}_2+\ln t+\ln F(\mathfrak{q})). \end{aligned} \tag{31}$$

This pseudo-hydrodynamic stream function solves the time independent equation

$$\begin{aligned} &\tilde{\Delta}_{(\mathfrak{p},\mathfrak{q})}G = \\ &\big(\tilde{\partial}_1 G+1\big)\big(\tilde{\partial}_2\tilde{\Delta}_{(\mathfrak{p},\mathfrak{q})}G\big)-\big(\tilde{\partial}_2 G-1\big)\big(\tilde{\partial}_1\tilde{\Delta}_{(\mathfrak{p},\mathfrak{q})}G\big). \end{aligned} \tag{32}$$

[3] Easily seen by permuting $k_1 = n+1-\sigma$ and $k_2 = \sigma$ and using formula (15).

The initial condition is replaced by a boundary condition at infinity: for $\tilde{z}_1 \to -\infty$, $\tilde{z}_2 \to -\infty$ holds

$$G(\tilde{z}_1,\tilde{z}_2) \simeq e^{\tilde{z}_1}+e^{\tilde{z}_2}. \tag{33}$$

As has been noted in [8], Equation (32) can be written in a form which is closer to (4) by introducing an auxiliary function

$$H(\tilde{z}_1,\tilde{z}_2) = G(\tilde{z}_1,\tilde{z}_2)+\tilde{z}_1-\tilde{z}_2. \tag{34}$$

We then obtain the following pseudo-hydrodynamic equation

$$\tilde{\Delta}_{(\mathfrak{p},\mathfrak{q})}H = \tilde{J}(H,\tilde{\Delta}_{(\mathfrak{p},\mathfrak{q})}H). \tag{35}$$

B. Parameter-dependence of the solutions and the limiting cases $\phi \to 0$ and $\phi \to \pi$

To get an idea about the properties of the solutions in dependence on the parameters we calculate the coefficients $G_\sigma^{(n)}(\phi,\eta)$ symbolically as functions of ϕ and ϕ up to $n = 8$, the coefficient $G_4^{(7)}(\phi,\eta)$ being represented on Fig. 2. From

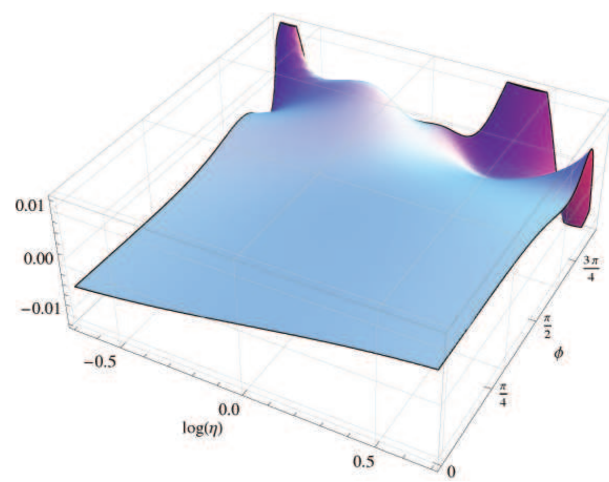

FIG. 2: Coefficient $G_4^{(7)}(\phi,\eta)$ in dependence on ϕ and η.

the study of the low-order coefficients we can infer that: for (i) ϕ close to 0 the dependence of the coefficients on both ϕ and η is very smooth, whereas for (ii) ϕ close to π the coefficients behave very irregularly. This suggests that the case $\phi \to 0$ may have a reduced non-linearity in comparison to the case $\phi \to \pi$. With this motivation in mind we are going to study the limits $\phi \to 0$ and $\phi \to \pi$.

In the case $\phi \to 0$ the short-time asymptotic solution is obtained in a straightforward fashion by putting $\phi = 0$. The Fourier coefficients of the solution can be calculated directly from the recursion relation (15) with $\phi = 0$. Indeed, the recursion relation (15) is well-defined because for $\phi = 0$ the Laplacian $((n+1-\sigma)/\sqrt{\eta}+\sqrt{\eta}\sigma)^2$ does not vanish.

The case $\phi \to \pi$ is much more delicate: the values of the coefficients depend on the procedure we choose to calculate them. Consider, for example, the behaviour of the coefficient

$G_1^{(2)}(\phi,\eta) = F_{+\ 1}^{(2,0)}(\mathfrak{p},\mathfrak{q})$ given by (18). As function of ϕ and η it is not continuous at $\eta = 1$, $\eta = 2$ and $\phi = \pi$. Indeed, if we fix first the angle $\phi = \pi$ we obtain the expression

$$G_1^{(2)}(\pi,\eta) = \frac{1}{2}\,\frac{\eta+1}{\eta-1}\,\frac{\eta}{\eta-2}, \tag{36}$$

which has poles[4] at $\eta = 1$ and $\eta = 2$. On the other hand, fixing $\eta = 1$ or $\eta = 2$ we will obtain finite values by considering the limits

$$\lim_{\phi\to\pi} G_1^{(2)}(\phi,1) = 0, \qquad \lim_{\phi\to\pi} G_1^{(2)}(\phi,2) = \frac{3}{4}. \tag{37}$$

Thus, the numerical values of the coefficients for given (η,ϕ) depend on the limiting procedure $\phi \to \pi$, $\eta' \to \eta$. Since we are mostly interested in the dependence of the solutions on the angle ϕ, it seems natural to consider the limit of the coefficients obtained by fixing the ratio of lengths η first and then letting $\phi \to \pi$. The actual implementation of this limit will be discussed in Section V and Appendix B.

Note that the $\phi \to 0$ (or $\phi \to \pi$) limit obtained by fixing η can also be obtained for the two-dimensional Euler equation (5) with real-valued initial conditions (6), see Appendix B. The relation to the so-called hydrostatic limit [18, 19, 20] remains to be clarified.

IV. CASE OF PARALLEL INITIAL MODES

In this section we shall consider numerical solutions for the case $\phi = 0$. As has been found in PMFB, the value of the exponent of the algebraic prefactor does not depend on the ratio of the lengths of the initial vector η. In the following for convenience we shall consider the case $\eta = 2$. We have calculated the numerical solution by using the recursion relation (15) with a precision of about 70 significant digits and with the resolution $N_{\max} = 2000$. All numerical calculations in this and the two following sections have been done using multiple-precision calculation packages MPUN90 package [24]) and MPFR [25].

We have studied asymptotic expansion of the coefficients $G^{(n)}(\sigma) = G(k_1,k_2)$ (where $k_1 = n+1-\sigma$ and $k_2 = \sigma$) along half-lines

$$|\boldsymbol{k}|(\cos\theta,\sin\theta) = (k_1,k_2) = m(p,q)$$

in the Fourier space (later we shall refer to them as *rational directions*) for $|\boldsymbol{k} \to \infty$. For this purpose we have used the asymptotic interpolation procedure of van der Hoeven [26], see also [27]. In Appendix A we briefly remind the reader the main principles of numerical analysis of asymptotic expansions.

We have chosen rational directions such that $p,q < 10$, where p and q are mutually prime numbers. This gives 55 different rational directions. First, after six stages of interpolation we have identified the leading order term and the two successive sub-leading order terms

$$G(k,\theta) \sim C(\theta)k^{-\alpha(\theta)}\mathrm{e}^{-\delta(\theta)k}, \tag{38}$$

Fig. 3 shows the discrepancy of the scaling exponent α, determined from the interpolation value at the sixth stage, from the value $5/2$ as a function of the angle θ. It is seen that the numerically determined $\alpha_{\rm num}$ differs from $5/2$ by less than 10^{-7}. We remind the reader that a theoretical argument has been given in PMFB suggesting that the true value of α does not depend on θ.

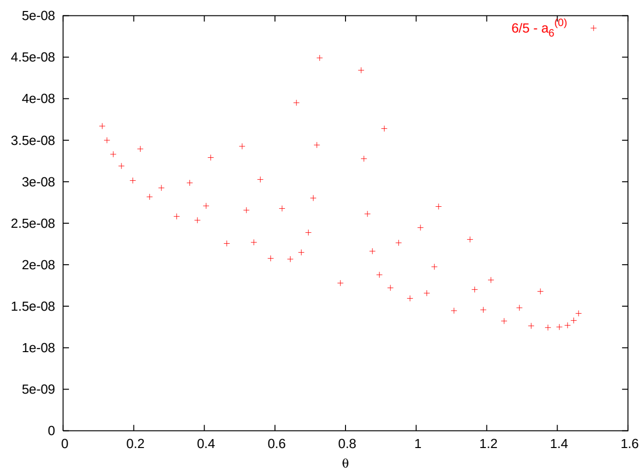

FIG. 3: Difference between the conjectured extrapolated value $3/\alpha = 6/5$ for $\alpha = 5/2$ and the numerical value $3/\alpha_{\rm num}$ obtained at the sixth stage of the asymptotic interpolation procedure.

To confirm that the algebraic prefactor exponent $\alpha = 5/2$ at the next step we have tried to identify higher-order corrections to (38). Indeed, if the higher-order corrections are known, we will be able to estimate the parameters $\delta(\theta)$, α and $C(\theta)$ more precisely. Furthermore, to obtain some information on the analytical structure of the solutions it is advantageous to know their asymptotic expansion at high orders.

We remark that at the sixth stage of the interpolation procedure the numerical data are of the form

$$g^{(6)}(m) = \frac{3}{\alpha} + \omega(m), \tag{39}$$

where m is the number of the point $m(p,q)$ on the line with rational direction (p,q). Continuing the procedure beyond the sixth stage we have first identified the nature of the remainder $\omega(m)$ by the procedure described in the Appendix A.

The identification has been performed for the coefficients corresponding to the most populated direction $(p,q) = (1,1)$, which is the direction with the most pronounced asymptotic behaviour. We have found that the correction is of the form $\omega(m) \sim 1/m^2$. This correction corresponds to the following

[4] Note that if we fix a rational η, then there exist n and σ such that the Fourier coefficient $F_\sigma^{(n)}(\pi,\eta)$ has a pole at η.

form of the asymptotic expansion of the Fourier coefficients

$$G(k,\theta) = C(\theta)k^{-\frac{5}{2}}\,\mathrm{e}^{-\delta(\theta)k}\left[1+\frac{a_1(\theta)}{k}+\frac{a_2(\theta)\ln k}{k^2}+O\left(\frac{1}{k^2}\right)\right]. \tag{40}$$

We note that the same asymptotic expansion is found for solutions of the linearised Euler equation considered in the short-time régime, see Appendix C.

It is very unlikely that the functional form of the asymptotic expansion depends on the direction (p,q) in the Fourier space. However, for large p and q the number of points on the half-lines with the rational direction (p,q) is not very high, and therefore the asymptoic behaviour may not be attained. Therefore the values of the coefficient $a_2(\theta)$ that we have measured using the asymptotic interpolation at thirteenth stage exhibit a quite large scatter. Due to this, for determining $\delta(\theta)$ and $C(\theta)$ (represented on Figs. 4 and 5) we have taken into account only the correction term $a_1(\theta)/k$.

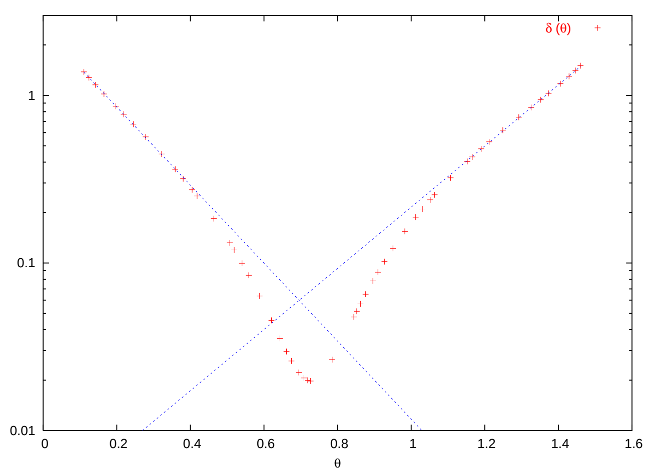

FIG. 4: Leading-order asymptotics: $\delta(\theta)$ in lin-log coordinates.

As we have stated before, averaged over all direction the absolute error in determining α is of the order 10^{-7}. However, restricting ourselves to the most populated rational directions, we can vastly increase the precision in determination of α. One possibility is for example to use one of the convergence acceleration algorithms such as the ρ-algorithm for the data at sixth stage (39). The choice of the algorithm is fixed through the form of the correction term being proportional to $1/m^2$, see [28]. For example, for the most populated direction $(1,1)$ the acceleration algorithm stabilises at $5/2$ with an error of the order 10^{-16} after 50 iterations.

V. CASE OF ANTI-PARALLEL INITIAL MODES

In this Section we study numerically the case of antiparallel modes $\phi \to \pi$: as we have seen before, this limit cannot be analysed directly by taking $\phi = \pi$. Instead, we fix η, taking the limit $\phi \to \pi$ of the recursion relation (15).

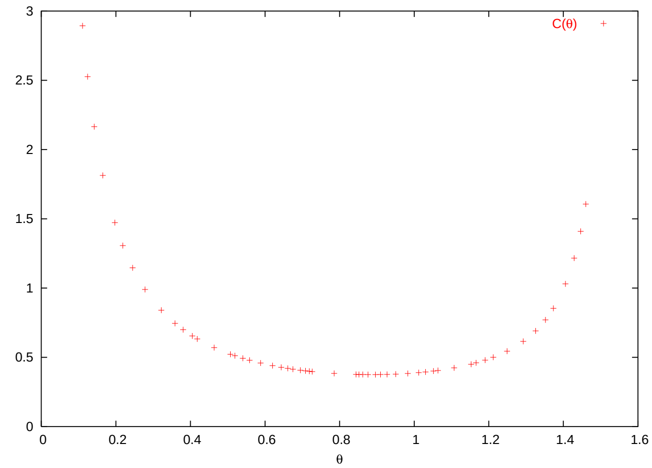

FIG. 5: Second subleading term $C(\theta)$.

Here we shall only consider the case[5] $\eta = 2$: the recursion relation is undefined for $n+1-\sigma = 2\sigma$ which we have to analyse separately. It turns out that both the sum on the right-hand side of (15) and the Laplacian $((n+1-\sigma)-2\sigma)^2$ vanish in such a way that their ratio yields a finite limit. The detailed calculation is not difficult but somewhat cumbersome; its details are given in Appendix B.

Numerical calculation of the coefficients reveals that the properties of the $\phi \to \pi$-limiting solution are very different from the properties of the solutions which have been observed before, e.g. from those of $\phi = 0$ or $\phi = \pi/2$. Remember that in these cases the Fourier coefficients had alternating sings which means that the part of the singular manifold being closest to the real domain is located "above" the point $(\pi, 0)$ in the real space.

In the present case the situation is more complicated. Firstly, it turns out that the Fourier coefficients $G^{(n)}(\sigma)$ vanish for $\sigma < n/3$ and thus the support of the solution in the Fourier space is bounded by the half-lines $k_1 = 1$ and $k_2 = (k_1-1)/2$. Secondly, as can be seen on Fig. 6, in the angular sector approximately between $k_2 = (k_1-1)/2$ and $k_2 = 3k_1$ the coefficients can have both positive and negative signs which are distributed in an interference pattern. And thirdly, we find that in the sector roughly between the half-lines $k_2 = 3k_1$ and $k_1 = 1$ the coefficients are strictly positive. This means that one part of the singular manifold is located over the point $(0,0)$, whereas the other part is found above some curve in the real space.

The geometry of the singular manifold can be studied by a two-dimensional generalisation of the BPH-method, discussed in [22], which is, however, out of the scope of the present article. Note that to study the type of the singularities it is sufficient to consider the sector with the positive coefficients, applying to them the asymptotic interpolation proce-

[5] It is not difficult to extend the calculation to other values of η, for example $\eta = 3$ or $\eta = 4$.

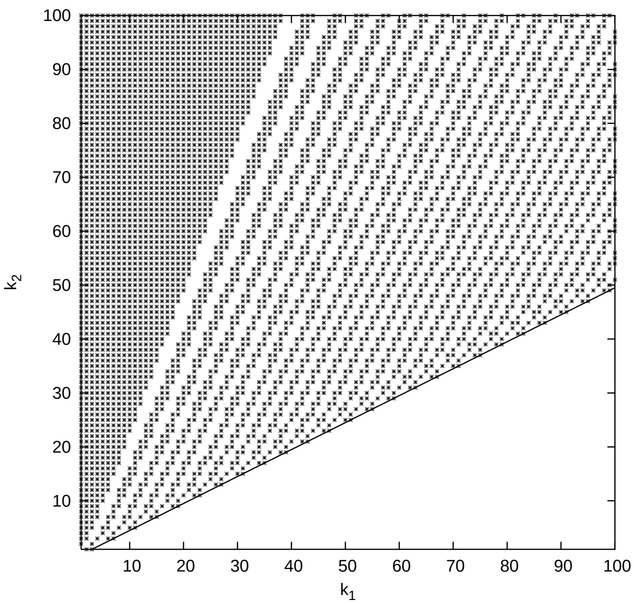

FIG. 6: Distribution of the signs of the Fourier coefficients in the in the limiting case $\phi \to \pi$. All Fourier coefficients vanish for $\sigma < n/3$.

dure used previously.

Here again for rational directions (p, q), $q/p \geq 3$ the leading order term and the two successive sub-leading order terms are

$$G(k, \theta) \sim C(\theta) k^{-\alpha(\theta)} \mathrm{e}^{-\delta(\theta) k}, \tag{41}$$

and the data at the sixth stage of the asymptotic interpolation procedure have the form

$$g^{(6)}(m) = \frac{3}{\alpha} + \omega(m). \tag{42}$$

Continuing the interpolation procedure up to the thirteenth stage we found that for slopes q/p bigger than 4 the correction can be determined very accurately and it is of the form $1/m^4$. The asymptotic expansion is just an expansion in $1/|\boldsymbol{k}|$

$$G(k, \theta) = C(\theta) k^{-3} \,\mathrm{e}^{-\delta(\theta) k} \left[1 + \frac{a_1(\theta)}{k} + \frac{a_2(\theta)}{k^2} + \frac{a_3(\theta)}{k^3} + O\left(\frac{1}{k^4}\right)\right]. \tag{43}$$

To increase the precision in the determination of α we use the ρ-algorithm to accelerate the convergence. We find that the constant $3/\alpha$ at sixth stage stabilises at the value 1 with more than 30 digits.

The value $\alpha = 3$ means that the vorticity diverges as $s^{-1/2}$ in the neighbourhood of the singular manifold. Note that for the two-dimensional Burgers equation with initial conditions that are given by the velocity field determined from the stream function (3), the derivative of velocity field diverges at the same rate. Also, the form of the asymptotic expansion (43) (expansion in inverse powers of k) is identical to the asymptotic expansion we would obtain for the Burgers equation. Thus, we conjecture that the complex singularities of the solutions of the Euler equation in the case $\phi \to \pi$ are of the same type as the corresponding singularities of the Burgers equation.

VI. SOME REMARKS ON THE GENERAL CASE OF $0 < \phi < \pi$

We have seen that the geometry as well as the nature of singularities are very different in the two limiting cases $\pi \to 0$ and $\phi \to \pi$: in the former case the singularities closest to the real domain are located over the symmetry point $(\pi, 0)$ whereas in the latter case they lie over the point $(0, 0)$ and a certain curve in the real domain. Furthermore, in these two cases the values of the scaling exponents ($\alpha = 5/2$ in the former and $\alpha = 3$ in the latter case) and the functional forms of the asymptotic expansions are different. Therefore, it is not surprising to find that the general case $0 < \phi < \pi$ exhibits strong intermediate asymptotics, making it difficult to determine the actual high-k behaviour of the Fourier coefficients. In particular this impedes a precise determination of the scaling exponent α and of the functional form of the asymptotic expansion.

The easiest way to see the interplay between the $\phi \to 0$ and $\phi \to \pi$ limiting behaviour is to look at the case when ϕ is very close to π. First of all, in this case the Fourier coefficients have non-vanishing values almost everywhere. An inspection of the sign distribution of the Fourier coefficients (Fig. 7) shows the existence of three different regions: the first

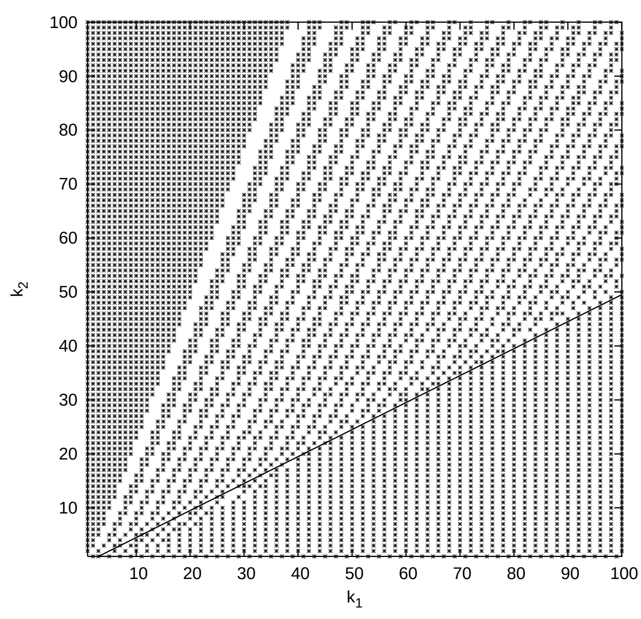

FIG. 7: Sign distribution for an angle close to π, here $\phi = 24/25\pi$.

region is roughly the one between the k_1 axis and the half-line with the slope $1/2$. The Fourier coefficients in this sector have alternating signs. The corresponding part of the singular manifold is therefore located over the point $(\pi, 0)$. Note that in the

case $\phi = \pi$ the Fourier coefficients in this sector vanish. The second region corresponds to the sector with the interference pattern in the case $\phi = \pi$. In the third region between the half-line with the slope 3 and the k_2-axis the Fourier coefficients are strictly positive. The corresponding part of the singular manifold is therefore located over the point $(0, 0)$. The singular manifold has a quite interesting geometry, however, its detailed study is out of the scope of the present article and will be presented elsewhere.

When the angle ϕ is decreased from π to 0 the region with alternating sings proliferates until all the Fourier coefficients become alternating in their signs. The threshold value of the angle is conjectured to be[6] $\phi = \arccos(-1/4)$.

In view of the complicated geometry of the singular manifold it is not surprising that a precise determination of the scaling exponent is very difficult and needs very high resolutions. One of the main factors reducing the precision in the determination of $\alpha(\phi)$ are the subdominant contributions which strongly depend on the directions in the Fourier space in which the Fourier coefficients are evaluated[7]. To increase the precision one would have to take into account these subdominant corrections.

However, with resolutions of order $N_{\max} = 2000$ we have not succeeded in determining the functional form of the asymptotic expansion of the Fourier coefficients and there could not obtain the numerical values of the scaling exponents with a precision comparable to the one found in the cases $\phi \to 0$ and $\phi \to \pi$. Therefore, a systematic study of the dependence of the scaling exponents on the angle is not feasible at the moment.

VII. CONCLUSIONS

In the present paper we have shown that, even for the simplest initial conditions with non-trivial dynamics, complex singularities of the solutions of the two-dimensional Euler equation can be of very different nature, corresponding to the values of the scaling exponent $\alpha = 5/2$ and $\alpha = 3$.

This has been achieved by identifying two particularly simple cases for which the exponents and therefore the type of the singularities can be determined very accurately. Furthermore, we were able to determine the subdominant terms. This increases the certainty in the numerical value of the scaling exponents far beyond the cases which have been analysed in PMFB.

In the limiting case $\phi \to 0$ it has been found that the value of the scaling exponent α coincides with the value of the scaling exponent for the solutions of the linearised Euler equation. Furthermore, we have numerical evidence that the structure of the asymptotic expansion in the Fourier space is the same in both cases. Thus, non-linearity is suppressed in this case, which constitutes an example of a dynamically nontrivial completely depleted flow.

Actually, the case $\phi \to 0$ can be treated by means of asymptotic expansions near the edges, i.e. the regions for which θ is either close to 0 or close to π. In particular, the value $\alpha = 5/2$ can be found both by Maple assisted calculations and asymptotic arguments[8].

In the case $\phi \to \pi$ the scaling exponent $\alpha = 3$ is the same as for solutions of the two-dimensional Burgers equation. Also in this case the structure of the asymptotic expansion coincides in both cases. This case is an example of a flow with a very strong nonlinearity, probably the strongest possible for the initial conditions of trigonometric polynomial type.

The difference between the two cases $\phi \to 0$ and $\phi \to \pi$ is reflected not only in the type of the singularities: we have found that in these two cases the geometry of complex singularities is completely different. In the case $\phi \to 0$ all (except one) Fourier coefficients of the solution have regularly alternating signs which in the physical space is expressed by the fact that the part of the singular manifold closest to the real domain is located "over" the symmetry point $(\pi, 0)$. In the case $\phi \to \pi$ only a part of the Fourier coefficients has regular signs being positive. The remaining coefficients do not have such regularity in the signs. The physical space counterpart means that the part of the singular manifold closest to the real domain is located partially "over" the point $(0, 0)$ and partially over some curve in the real domain.

For intermediate angles ϕ lying between 0 and π, for example in the SOC case $\phi = \pi/2$ we have not been able to determine the subdominant terms in the asymptotic expansion. Therefore, the uncertainty in the exact value of the scaling exponent, in particular in dependence on the angle ϕ is quite high.

What do our findings mean in consideration of the question of well-posedeness of the three-dimensional Euler equation? Firstly we note that the short-time asymptotic analysis of the kind presented here is easily extendable to three dimensions. Some preliminary results in this direction have been obtain in [22] confirming non-universality of the nature of complex singularities in three dimensions as well. Secondly one very important unresolved issue remains: does the nature of complex singularities change when we go beyond the short-time asymptotics? Although some indications have been given in PMFB that the nature of singularities does not change we have no definitive answer yet.

We expect that the extent to which the non-linearity is depleted is strongly dependent on the initial conditions which we can also view as a dependence on the large-scale structures in the flow. We cannot exclude that even if solutions of the three-dimensional Euler equation generically stay regular,

[6] This threshold value applies only to the case $\eta = 2$. In the more general case its value is $\arccos(-1/2\eta)$. The threshold values of ϕ have been estimated using symbolically calculated expressions for the Fourier coefficients.

[7] This introduces an artificial dependence of the scaling coefficients obtained numerically on the direction θ in the Fourier space. As we have mentioned before, it can be shown analytically that the value of the scaling exponent does not depend on θ.

[8] A. Gilbert, private communication.

there exist initial conditions – such as the Kida–Pelz – with very strong non-linearity which could produce a finite-time blow up.

Finally we note that the algebraic approach of the type used here can be extended to the viscous case. The short-time asymptotic régime of the inviscid case corresponds in the Navier–Stokes case to the asymptotic régime of small Reynolds numbers.

Acknowledgments

The author would like to thank U. Frisch for encouragement, support and many stimulating insights. A. Gilbert is acknowledged for his constant interest in this work. The author also thanks T. Matsumoto, J. Bec, D. Mitra and D. Vincenzi for many interesting discussions.

APPENDIX A: ASYMPTOTIC EXPANSIONS OF FOURIER/TAYLOR COEFFICIENTS

Let us consider a two-dimensional array $\hat{G}(k_1, k_2)$ with $k_1, k_2 \geq 0$. One possibility to study the asymptotic behaviour of $\hat{G}(k_1, k_2)$ at infinity is to proceed by simple reduction, analysing the behaviour of the one-dimensional arrays[9]

$$\hat{g}_{(p,q)}(m) = \hat{G}(mp, mq), \tag{A.1}$$

for relatively prime positive numbers p, q. An asymptotic expansion of $\hat{g}_{(p,q)}(m)$ for $m \to \infty$ yields then an asymptotic expansion of $\hat{G}(k_1, k_2)$ in polar coordinates $(k_1, k_2) = |\boldsymbol{k}|(\cos\theta, \sin\theta)$ for $|\boldsymbol{k}|$. Note that other approaches have been suggested in [26]. They are not discussed here.

The asymptotic expansion of $\hat{g}_{(p,q)}(m)$ is found by applying the asymptotic interpolation procedure [26]. Due to the presence of complex singularities the leading order and the two first sub-leading order contributions are

$$\hat{g}_{(p,q)}(m) \simeq C_{(p,q)} m^{-\alpha} \mathrm{e}^{-\delta_{(p,q)} m}, \tag{A.2}$$

see also [8, 22].

The main idea of the asymptotic interpolation procedure is to find for $\hat{g}_{(p,q)}$ such sequence transformations – stages of the interpolation procedure – which would "strip off" the sequence $\hat{g}_{(p,q)}$ the leading order terms $e^{-\delta_{(p,q)} m}$ and $m^{-\alpha}$ without knowing the precise value of the constants $\delta_{(p,q)}$ and α. One possible choice of transformations is the one which has been given in [27]:

$$\mathbf{SR}\,,\ -\mathbf{D}\,,\ \mathbf{I}\,,\ \mathbf{D}\,,\ \mathbf{D}\,,\ \mathbf{D}, \tag{A.3}$$

where the sequence transformations are: $\mathbf{I}$ taking the inverse $\hat{g}(m) \to 1/\hat{g}(m)$, $\mathbf{SR}$ taking the second ratio $\hat{g}(m) \to \hat{g}(m)\hat{g}(m-2)/(\hat{g}(m-1))^2$ and $\mathbf{D}$ taking the difference $\hat{g}(m) \to \hat{g}(m) - \hat{g}(m-1)$. After applying these sequence transformations the leading order of the obtained sequence together with corrections will be $3/\alpha + \omega(m)$.

The main problem in studying the asymptotics of the Fourier coeffcients is the correct identification of the correction term $\omega(m)$. Indeed, once its form is known we can draw conclusions about the further sub-leading order terms in the asymptotic expansion (A.2) of $\hat{g}(m)$. If $\omega(m)$ is to the leading order of the type $m^{-\gamma}$, as in the case of the limits $\phi \to 0$ and $\phi \to \pi$, we can identify the constant γ by applying to the data after (A.3) the following transformations

$$\mathbf{D}\,,\ \mathbf{I}\,,\ \mathbf{R}\,,\ \mathbf{D}\,,\ \mathbf{I}\,,\ \mathbf{D}\,,\ \mathbf{D}. \tag{A.4}$$

The leading order term after these transformations will have the form $2/(\gamma + 1)$. In the cases $\phi \to 0$ and $\phi \to \pi$ the constant γ can be identified as $\gamma = 2$ and $\gamma = 4$ respectively.

[9] The array $\hat{g}(mp, mq)$ is closely connected to the so-called diagonal of a double Taylor series, see [29].

APPENDIX B: PARALLEL AND ANTI-PARALLEL MODES LIMITS OF THE TWO-DIMENSIONAL EULER EQUATION

1. Two-dimensional Euler equation in the $\phi \to 0$ limit

We begin by considering the two-dimensional Euler equation (9) with initial conditions (6). Putting formally $\phi = 0$ (or $\phi = \pi$) we obtain

$$\begin{aligned} &\partial_t \big(\frac{1}{\sqrt{\eta}}\,\overline{\partial}_1 \pm \sqrt{\eta}\,\overline{\partial}_2\big)^2 \overline{\psi}_{\rm form} = \\ &\overline{J}\Big(\overline{\psi}_{\rm form}, \big(\frac{1}{\sqrt{\eta}}\,\overline{\partial}_1 \pm \sqrt{\eta}\,\overline{\partial}_2\big)^2 \overline{\psi}_{\rm form}\Big), \end{aligned} \tag{B.1}$$

where the initial condition remain unchanged and the signs $+$ and $-$ corresponds to the cases $\phi = 0$ and $\phi = \pi$. Unfortunately this equation is ill-posed, because, contrarily to the Laplacian $\Delta_{(\phi,\eta)}$ in the case $0 < \phi < \pi$, the operator $((1/\sqrt{\eta})\,\overline{\partial}_1 \pm \sqrt{\eta}\,\overline{\partial}_2)^2$ has a non-vanishing kernel.

Here we show that if we first fix η and then let $\phi \to 0$ we obtain a well-defined equation for the $\phi \to 0$ limit of (9). Consider the two-dimensional Euler equation in the Fourier space

$$\begin{aligned} &\partial_t \hat{\psi}(k_1, k_2) = \\ &-\frac{1}{|\mathfrak{p}k_1 + \mathfrak{q}k_2|^2} \sum_{\boldsymbol{l}+\boldsymbol{l}'=\boldsymbol{k}} \boldsymbol{l} \wedge \boldsymbol{l}' \hat{\psi}(l_1, l_2) |\mathfrak{p}l_1' + \mathfrak{q}l_2'|^2 \hat{\psi}(l_1', l_2'). \end{aligned} \tag{B.2}$$

Clearly, for $\mathfrak{p} = (1, 0)$ and $\mathfrak{q} = (\eta, 0)$, which corresponds to the limit $\phi \to 0$ the denominator on the right-hand side vanishes. However, as we shall see now, the numerator also vanishes in such a way that the ratio of the both gives a finite number in the limit $\phi \to 0$.

We rearrange the sum in such a way that the term $\hat{\psi}(l_1, l_2)\hat{\psi}(l_1', l_2')$ appears only once for every $\boldsymbol{l}$. Thus, we obtain terms of the form

$$\boldsymbol{l} \wedge \boldsymbol{l}' \left(|\mathfrak{p}l_1' + \mathfrak{q}l_2'|^2 - |\mathfrak{p}l_1 + \mathfrak{q}l_2|^2\right) \hat{\psi}(l_1, l_2)\hat{\psi}(l_1', l_2'), \tag{B.3}$$

where $\boldsymbol{l}+\boldsymbol{l}'=\boldsymbol{k}$. Noting that

$$|\mathfrak{p}l_1'+\mathfrak{q}l_2'|^2-|\mathfrak{p}l_1+\mathfrak{q}l_2|^2= \Big(\frac{1}{\sqrt{\eta}}k_1+\sqrt{\eta}k_2\Big)^2-2k_1k_2(1-\cos\phi)-2l_1\Big(\frac{1}{\eta}k_1+k_2\Big)+ 2l_1k_2(1-\cos\phi)-2l_2(k_1+\eta k_2)+2l_2k_1(1-\cos\phi), \tag{B.4}$$

and

$$|\mathfrak{p}k_1+\mathfrak{q}k_2|^2=\Big(\frac{1}{\sqrt{\eta}}k_1+\sqrt{\eta}k_2\Big)^2-2k_1k_2(1-\cos\phi), \tag{B.5}$$

we see that when $k_1+\eta k_2=0$, it follows

$$\frac{|\mathfrak{p}l_1'+\mathfrak{q}l_2'|^2-|\mathfrak{p}l_1+\mathfrak{q}l_2|^2}{|\mathfrak{p}k_1+\mathfrak{q}k_2|^2}=\frac{k_1k_2-l_1k_2-l_2k_1}{k_1k_2}\neq 0, \tag{B.6}$$

for $k_1,k_2\neq 0$. The only case where the above remark does not apply is when there exists an $\boldsymbol{l}$ such that $2\boldsymbol{l}=\boldsymbol{k}$, in which case we obtain

$$\frac{|\mathfrak{p}l_1'+\mathfrak{q}l_2'|^2}{|\mathfrak{p}k_1+\mathfrak{q}k_2|^2}=\frac{1}{4}. \tag{B.7}$$

Thus, the expression on the right-hand side of (B.2) is well-defined in the limit $\phi\to 0$. The analytic properties of this limiting solution and relation to solutions of the so-called hydrostatic equation certainly merit a further study. We also note that a similar limiting solution can be constructed for the two-dimensional Navier–Stokes equation.

2. Short-time recursion relation in the $\phi\to\pi$ limit

Let us consider the limiting form of the recursion relation (15) in the case $\eta=2$ and $\phi\to\pi$. Firstly, we remark that in the sum on the right-hand-side of this recursion relation the terms containing the product $G_\tau^{(m)}G_{\sigma-\tau}^{(n-m-1)}$ appear twice: firstly for $(m,\,\tau)$ and secondly for $(n-m-1,\,\sigma-\tau)$. Thus we can regroup the recursion relation in such a way as that the product $G_\tau^{(m)}G_{\sigma-\tau}^{(n-m-1)}$ appears in only one summand, which has the form

$$K_{(\mathfrak{p},\mathfrak{q})}(n,\sigma|m,\tau)\times \big[(m+1)(\sigma-\tau)-(n-m)\tau\big]G_\tau^{(m)}G_{\sigma-\tau}^{(n-m-1)}, \tag{B.8}$$

where

$$K_{(\mathfrak{p},\mathfrak{q})}(n,\sigma|m,\tau)=|\big((n-m)-(\sigma-\tau)\big)\mathfrak{p}+(\sigma-\tau)\mathfrak{q}|^2- |(m+1-\tau)\mathfrak{p}+\tau\mathfrak{q}|^2 \tag{B.9}$$

Secondly, since for $\eta=2$ the recursion relation (15) becomes undefined only when $n+1-\sigma=2\sigma$, we concentrate on the $\phi\to\pi$-limit of the recursion relation only for $n=3\sigma-1$. As we shall see now, when $\phi\to\pi$

$$\frac{1}{|(n+1-\sigma)\mathfrak{p}+\sigma\mathfrak{q}|^2}K_{(\mathfrak{p},\mathfrak{q})}(n,\sigma|m,\tau)=\frac{1}{2\sigma^2} \Big\{\big((n-m)-(\sigma-\tau)\big)(\sigma-\tau)-(m+1-\tau)\tau\Big\}, \tag{B.10}$$

and has thus a finite value.

Indeed, assume that n_1, n_1', n_2 and n_2' are integers such that $n_1+n_1'=2\sigma$ and $n_2+n_2'=\sigma$. Then it holds

$$|n_1\mathfrak{p}+n_2\mathfrak{q}|^2-|n_1'\mathfrak{p}+n_2'\mathfrak{q}|^2=4(n_1n_2-n_1'n_2')(1+\cos\phi), \tag{B.11}$$

where we have used the expressions $\mathfrak{p}=(1,0)$ and $\mathfrak{q}=2(\cos\phi,\sin\phi)$. Similarly we obtain

$$|(n+1-\sigma)\mathfrak{p}+\sigma\mathfrak{q}|^2=8\sigma^2(1+\cos\phi), \tag{B.12}$$

and therefore the term $1+\cos\phi$ is cancelled.

APPENDIX C: PERTURBATIVE STUDY OF THE SHORT-TIME ASYMPTOTICS

Here we present a simple linear model which turns out to have the same asymptotic structure as the solution of the short-time Euler equation in the limit $\phi=0$.

As we have noted in Section II B for small values of $\varepsilon=\eta-1$ we can apply a perturbative ansatz to Equation (32), which will give us to the first sub-leading order

$$(1-\mathrm{e}^{\tilde z_2})\check\partial_1\tilde\Delta_{(\phi,1)}G_1+(1+\mathrm{e}^{\tilde z_1})\tilde\partial_2\tilde\Delta_{(\phi,1)}G_1-\tilde\Delta_{(\phi,1)}G_1+ \mathrm{e}^{\tilde z_2}(\tilde\partial_1G_1)-\mathrm{e}^{\tilde z_1}(\check\partial_2G_1)=-2\,\mathrm{e}^{\tilde z_1}\mathrm{e}^{\tilde z_2}. \tag{C.1}$$

Neglecting the source term and the terms $\mathrm{e}^{\tilde z_2}(\tilde\partial_1G_1)-\mathrm{e}^{\tilde z_1}(\check\partial_2G_1)$ we obtain the passive scalar model which has been introduced in [8]. For this model the singular manifold can be described analytically, and furthermore, we can determine analytically the type of singularities of the field $\tilde\Delta_1G_1$, which turn out to be poles. So far we have not been able to handle Equation (C.1) satisfactorily by analytic tools. Therefore we have studied solutions of this equation by using high-precision numerics (for these calculations we have used a precision of about 70 digits and a resolution $N_{\max}=3000$.).

We have found that the presence (or absence) of the source term and of the term $\mathrm{e}^{\tilde z_2}(\tilde\partial_1G_1)-\mathrm{e}^{\tilde z_1}(\check\partial_2G_1)$ changes neither the location of the singularities nor the value of algebraic prefactor exponent α. Using the asymptotic interpolation procedure of we were able to determine the asymptotic expansion of the solution in the Fourier space, which has the same form as the asymptotic expansion (40) of the short-time Euler equation in the limit $\phi\to 0$.

Note that the form of singular manifold and the scaling exponent $5/2$ (or 1 in the physical space) do not depend on the parameter ϕ. Thus, the perturbative expansion predicts a universal exponent and thus, gives wrong predictions when $\phi\neq 0$. Still, in the case $\phi=0$ it gives the correct form of the asymptotic expansion.

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